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# REPRESENTATION OF A QUOTIENT OF SOLUTIONS OF A FOUR-TERM LINEAR RECURRENCE RELATION IN THE FORM OF A BRANCHED CONTINUED FRACTION

The quotient of two linearly independent solutions of a four-term linear recurrence relation is represented in the form of a branched continued fraction with two branches of branching by analogous with continued fractions. Formulas of partial numerators and partial denominators of this branched continued fraction are obtained. The solutions of the recurrence relation are canonic numerators and canonic denominators of $\mathcal{B}$-figured approximants. Two types of figured approximants $\mathcal{A}$-figured and $\mathcal{B}$-figured are often used. A $n$th $\mathcal{A}$-figured approximant of the branched continued fraction is obtained by adding a next partial quotient to the $(n-1)$th $\mathcal{A}$-figured approximant. A $n$th $\mathcal{B}$-figured approximant of the branched continued fraction is a branched continued fraction that is a part of it and contains all those elements that have a sum of indexes less than or equal to $n$. $\mathcal{A}$-figured approximants are widely used in proving of formulas of canonical numerators and canonical denominators in a form of a determinant, $\mathcal{B}$-figured approximants are used in solving the problem of corresponding between multiple power series and branched continued fractions. A branched continued fraction of the general form cannot be transformed into a constructed branched continued fraction. For calculating canonical numerators and canonical denominators of a branched continued fraction with $N$ branches of branching, $N > 1$, the linear recurrent relations do not hold. $\mathcal{B}$-figured convergence of the constructed fraction in a case when coefficients of the recurrence relation are real positive numbers is investigated.

*Key words and phrases:* branched continued fraction, four-term recurrence relation.

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## INTRODUCTION

It is well known that the general solution of a linear homogeneous recurrence relation of second order: $y_n = b_n y_{n-1} + a_n y_{n-2}$, $n = 1, 2, \ldots$, where the $a_n, b_n$, $n \geq 1$, are complex numbers, can be represented in a form of a linear combination of two linearly independent solutions

$$y^{(1)} = \left(1, 0, y_1^{(1)}, y_2^{(1)}, \ldots\right), \; y^{(2)} = \left(0, 1, y_1^{(2)}, y_2^{(2)}, \ldots\right).$$

These solutions are, respectively, canonical numerators and canonical denominators of approximants of the continued fractions [15, 18, 19]

$$\mathop{\mathrm{D}}_{k=1}^{\infty} \frac{a_k}{b_k}.$$

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In this paper, an analogous idea for a four-term linear recurrence relation

$$y_n = c_n y_{n-1} + b_n y_{n-2} + a_n y_{n-3}, \tag{1}$$

where the $a_n, b_n, c_n,\ n \geq 2$, are complex numbers, is considered.

Different constructions of multidimensional generalizations arise as a result of considering the $N$-term recurrent relation, $N > 1$, [8, 12, 18]. They are widely used for compatible approximations, for representation of solutions of algebraic equations, etc. The formulas of the elements of these fractions were not obtained, in general, except for the Furshtenau's two-dimensional generalization of continued fractions [14]. B. V. Krukowski has proved the theorem of convergence of these fractions [16].

This investigation leads to branched continued fractions (BCF) that are a multidimensional generalization of continued fractions. Thus, BCF of the general form are under consideration [7, 9, 11, 21]. Also, the different forms of BCF exist, in particular, BCF of the special form [1, 3–6, 10, 13], two-dimentional continued fractions [2, 17, 20], etc. The different constructions of their approximants [7] and, respectively, the different types of convergence appear in the considering of different mathematical problems.

Let

$$\mathcal{I} = \left\{ i(k) = (i_1, i_2, \ldots, i_k)\ : 1 \leq i_p \leq 2,\ p = \overline{1,k},\ k \geq 1 \right\}$$

be the set of multiindices. Let us introduse an order relation $\prec$ on the set $\mathcal{I}$ for $i(p) \in \mathcal{I}$ and $j(q), j(p) \in \mathcal{I}$, where $j(s) = (j_1, j_2, \ldots, j_s)$, $s \in \mathbb{N}$:

1) $i(p) \prec j(q)$, if $p < q$;

2) $i(p) \prec j(p)$, if $i_1 < j_1$;

3) $i(p) \prec j(p)$, if exists $r$, $1 \leq r < p$, such that $i_k = j_k$, $k = \overline{1,r}$, $i_{r+1} < j_{r+1}$.

Let we have sequenses of complex numbers $\left\{\xi_{i(k)}\right\}$, $\left\{\eta_{i(k)}\right\}$, where $i(k) \in \mathcal{I}$, then

$$\sum_{i_1=1}^{2} \cfrac{\xi_{i(1)}}{\eta_{i(1)} + \sum\limits_{i_2=1}^{2} \cfrac{\xi_{i(2)}}{\eta_{i(2)} + \ddots}} = \sum_{i_1=1}^{2} \frac{\xi_{i(1)}}{\eta_{i(1)}} + \sum_{i_2=1}^{2} \frac{\xi_{i(2)}}{\eta_{i(2)}} + \ldots = \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{2} \frac{\xi_{i(k)}}{\eta_{i(k)}} \tag{2}$$

be a general branched continued fraction with two branches of branching with complex elements.

A $n$th approximant of the BCF (2) is a finite BFC of the form

$$f_n = \mathop{\mathrm{D}}_{k=1}^{n} \sum_{i_k=1}^{2} \frac{\xi_{i(k)}}{\eta_{i(k)}},\ n \geq 1. \tag{3}$$

The continued fraction

$$\frac{\xi_{i(1)}}{\eta_{i(1)}} + \frac{\xi_{i(2)}}{\eta_{i(2)}} + \ldots + \frac{\xi_{i(k)}}{\eta_{i(k)}} + \ldots \tag{4}$$

is called a $(i_1, i_2, \ldots, i_k, \ldots)$ branch of the BCF (2). Let us fix $i(n) \in \mathcal{I}$, then a $(i_1, i_2, \ldots, i_n)$ branch be a finite branch of the BCF (3).

Length of a finite $(i_1, i_2, \ldots, i_n)$ branch of the BCF (3) is a number of partial quotient of the $n$th approximant of the continued fraction (4).

Each branch in the finite BFC (3) has length equal $n$. A figured approximant of the BFC (2) is a BFC that is a part of (2) and has at least two branches with nonequal length. Two types of figured approximants are often used. In particular, $\mathcal{A}$-figured approximants are widely used in proving of formulas of canonical numerators and canonical denominators in a form of a determinant [7], $\mathcal{B}$-figured approximants are used in solving the problem of corresponding betweeen a multiple power series and a BFC.

Let $\dfrac{a}{b} \equiv \dfrac{c}{d}$ denotes that $a = c, b = d$.

A $n$th $\mathcal{B}$-figured approximanth of (2) is a BCF

$$\widehat{f}_n = \mathop{\mathrm{D}}_{k=1}^{n} \sum_{i_k=1}^{2} \frac{\xi^*_{i(k)}}{\eta^*_{i(k)}}, \ n \geq 1, \tag{5}$$

where

$$\frac{\xi^*_{i(k)}}{\eta^*_{i(k)}} \equiv \begin{cases} \dfrac{\xi_{i(k)}}{\eta_{i(k)}}, & \text{if } i_1 + i_2 + \ldots + i_k \leq n; \\ \dfrac{0}{1}, & \text{if } i_1 + i_2 + \ldots + i_k > n. \end{cases}$$

The BCF (2) converges ($\mathcal{B}$-figured converges), if the finite limit of its sequence of approximants $f_n$ ($\mathcal{B}$-figured approximants $\widehat{f}_n$) exists.

The canonical numerator $A_n$ and the canonical denominator $B_n$ of the $\mathcal{B}$-figured approximant $\widehat{f}_n$ are, respectively, the numerator and the denominator of a calculated BCF (5), $\widehat{f}_n = A_n/B_n$. In calculating we use the following algorithm [7]

$$\frac{A_n}{B_n} \equiv \sum_{i_1=1}^{2} \frac{\xi^*_{i(1)}\eta'_{i(1)}}{\eta^*_{i(1)}\eta'_{i(1)} + \xi'_{i(1)}}, n \geq 1, \tag{6}$$

and

$$\frac{\xi'_{i(m)}}{\eta'_{i(m)}} \equiv \sum_{i_{m+1}=1}^{2} \frac{\xi^*_{i(m+1)}\eta'_{i(m+1)}}{\eta^*_{i(m+1)}\eta'_{i(m+1)} + \xi'_{i(m+1)}}, \ i(m) \in \mathcal{I}, \ m = n-1, n-2, \ldots, 1; \ n \geq 2, \tag{7}$$

where

$$\xi'_{i(n)} = 0, \ \eta'_{i(n)} = 1, \ i_p = \overline{1,2}, \ p = \overline{1,n}, n \geq 1. \tag{8}$$

The algorithm (6)–(8) is equivalent to the gradual algorithm of calculation of the BCF (5) without any reductions in the process.

## 1 SECTION WITH RESULTS

Let the $y^{(1)} = (1, 0, b_1, y_2^{(1)}, y_3^{(1)}, \ldots)$, $y^{(2)} = (0, 1, c_1, y_2^{(2)}, y_3^{(2)}, \ldots)$, be the two solutions of equation (1), where the $b_1$, $c_1$ are complex numbers. These solutions yield all three linear independent solutions of (1), for example,

$$y^{(1)} = (1, 0, 0, y_2^{(1)}, y_3^{(1)}, \ldots), \ y^{(2)} = (0, 1, 0, y_2^{(2)}, y_3^{(2)}, \ldots), \ y^{(3)} = (1, 0, 1, y_2^{(3)}, y_3^{(3)}, \ldots).$$

Put $A_k = y_k^{(1)},\ B_k = y_k^{(2)},\ k = -1, 0, 1, \ldots,$ where

$$\begin{aligned} A_n &= c_n A_{n-1} + b_n A_{n-2} + a_n A_{n-3},\ n = 2, 3 \ldots, \\ B_n &= c_n B_{n-1} + b_n B_{n-2} + a_n B_{n-3},\ n = 2, 3 \ldots, \end{aligned} \tag{9}$$

and

$$A_{-1} = 1,\ A_0 = 0,\ A_1 = b_1,\ B_{-1} = 0,\ B_0 = 1,\ B_1 = c_1. \tag{10}$$

By analogous with a continued fraction let us construct the BCF such that each its $n$th $\mathcal{B}$-figured approximant equals $A_n/B_n$, $n \geq 1$.

If $n = 1$ then $A_1/B_1 = b_1/c_1$. For $n = 2$ we have $A_2/B_2 = b_1/(c_1 + b_2 c_2^{-1}) + a_1/(c_2 c_1 + b_2)$. If $n \geq 3$ we replace $n$ by $n-1$ in (9) and put the obtained value $A_{n-1}$ in (9), we get

$$A_n = \gamma_{n-1}^{(n)} A_{n-2} + \beta_{n-1}^{(n)} A_{n-3} + \alpha_{n-1}^{(n)} A_{n-4}, \tag{11}$$

where $\gamma_{n-1}^{(n)} = c_{n-1}c_n + b_n$, $\beta_{n-1}^{(n)} = b_{n-1}c_n + a_n$, $\alpha_{n-1}^{(n)} = a_{n-1}c_n$. Next, if $n \geq 4$, by substituting $n-2$ for $n$ in (9) and putting obtained $A_{n-2}$ in (11), we get a new formula for $A_n$, etc. If $n \geq r+2$, after $(n-r)$ steps we have

$$A_n = \gamma_r^{(n)} A_{r-1} + \beta_r^{(n)} A_{r-2} + \alpha_r^{(n)} A_{r-3}, \tag{12}$$

where

$$\gamma_r^{(n)} = c_r \gamma_{r+1}^{(n)} + \beta_{r+1}^{(n)},\ \beta_r^{(n)} = b_r \gamma_{r+1}^{(n)} + \alpha_{r+1}^{(n)},\ \alpha_r^{(n)} = a_r \gamma_{r+1}^{(n)}, \tag{13}$$

$r = n-1, n-2, \ldots, 2$, and $\gamma_n^{(n)} = c_n$, $\beta_n^{(n)} = b_n$, $\alpha_n^{(n)} = a_n$.

An analogous relation holds for $B_n$

$$B_n = \gamma_r^{(n)} B_{r-1} + \beta_r^{(n)} B_{r-2} + \alpha_r^{(n)} B_{r-3}, \tag{14}$$

where $\gamma_r^{(n)}, \beta_r^{(n)}, \alpha_r^{(n)}, r = n-1, n-2, \ldots, 2$, are defined by (13) and $\gamma_n^{(n)} = c_n$, $\beta_n^{(n)} = b_n$, $\alpha_n^{(n)} = a_n$, with initial conditions from (10).

Let us introduse the following notation

$$\begin{aligned} c'_k &= c_k c_{k-1} + b_k,\ k = \overline{2,n};\ n \geq 2; \\ b'_k &= b_k c_{k-2} + a_k,\ k = \overline{3,n};\ n \geq 3; \\ a'_k &= a_k c_{k-3},\ k = \overline{4,n};\ n \geq 4; \end{aligned} \tag{15}$$

and

$$w_j^{(n)} = \frac{\beta_j^{(n)}}{\gamma_j^{(n)}},\ j = \overline{1,n}, \qquad v_j^{(n)} = \frac{c_{j-2}\beta_j^{(n)} + \alpha_j^{(n)}}{\gamma_j^{(n)}},\ j = \overline{3,n},\ n \geq 3. \tag{16}$$

Combining this with the initial conditions (10) and relations (12)–(14), for $r = 2$, we obtain

$$\frac{A_n}{B_n} = \frac{\gamma_2^{(n)} A_1 + \beta_2^{(n)} A_0 + \alpha_2^{(n)} A_{-1}}{\gamma_2^{(n)} B_1 + \beta_2^{(n)} B_0 + \alpha_2^{(n)} B_{-1}} = \frac{\gamma_2^{(n)} b_1 + \alpha_2^{(n)}}{\gamma_2^{(n)} c_1 + \beta_2^{(n)}} = \frac{\beta_1^{(n)}}{\gamma_1^{(n)}} = w_1^{(n)},\ n \geq 3.$$

Using the denoting (15) and (16) we get

$$w_1^{(n)} = \frac{b_1}{c_1 + w_2^{(n)}} + \frac{a_2 \gamma_3^{(n)}}{c_1(c_2\gamma_3^{(n)} + \beta_3^{(n)}) + b_2\gamma_3^{(n)} + \alpha_3^{(n)}} = \frac{b_1}{c_1 + w_2^{(n)}} + \frac{a_2}{c'_2 + v_3^{(n)}}.$$

Let us prove the recurrent formulas for $w_k^{(n)}$, $k=\overline{2,n-2}$, $n\geq 4$, $v_k^{(n)}$, $k=\overline{3,n-2}$, $n\geq 5$. We obtain

$$w_k^{(n)}=\frac{\beta_k^{(n)}}{\gamma_k^{(n)}}=\frac{b_k\gamma_{k+1}^{(n)}+\alpha_{k+1}^{(n)}}{c_k\gamma_{k+1}^{(n)}+\beta_{k+1}^{(n)}}=\frac{b_k}{c_k+w_{k+1}^{(n)}}+\frac{a_{k+1}}{c'_{k+1}+v_{k+2}^{(n)}}. \tag{17}$$

Analogously

$$v_k^{(n)}=\frac{c_{k-2}\beta_k^{(n)}+\alpha_k^{(n)}}{\gamma_k^{(n)}}=\frac{c_{k-2}(b_k\gamma_{k+1}^{(n)}+\alpha_{k+1}^{(n)})+a_k\gamma_{k+1}^{(n)}}{c_k\gamma_{k+1}^{(n)}+\beta_{k+1}^{(n)}}=\frac{b'_k}{c_k+w_{k+1}^{(n)}}+\frac{a'_{k+1}}{c'_{k+1}+v_{k+2}^{(n)}}. \tag{18}$$

Let us now consider the case $k=n-1$

$$w_{n-1}^{(n)}=\frac{b_{n-1}}{c_{n-1}+\dfrac{b_n}{c_n}}+\frac{a_n}{c'_n},\ n\geq 2,\ v_{n-1}^{(n)}=\frac{b'_{n-1}}{c_{n-1}+\dfrac{b_n}{c_n}}+\frac{a'_n}{c'_n},\ n\geq 4. \tag{19}$$

If we put $w_n^{(n)}=\dfrac{b_n}{c_n}$, $v_n^{(n)}=\dfrac{b'_n}{c_n}$, $w_{n+1}^{(n)}=v_{n+1}^{(n)}=0$, $w_{n+2}^{(n)}=v_{n+2}^{(n)}=\infty$ we have that recurrent formulas (17), (18) hold for $k=n-1,n$, as well.

Consider the BCF (2), where

$$\xi_1=b_1,\ \xi_2=a_2, \tag{20}$$

and for all $i(k)\in\mathcal{I}, k\geq 2$

$$\xi_{i(k)}=\begin{cases} b_{i_1+i_2+\ldots+i_k}, & \text{if } i_{k-1}=i_k=1;\\ b'_{i_1+i_2+\ldots+i_k}, & \text{if } i_{k-1}=2, i_k=1;\\ a_{i_1+i_2+\ldots+i_k}, & \text{if } i_{k-1}=1, i_k=2;\\ a'_{i_1+i_2+\ldots+i_k}, & \text{if } i_{k-1}=2, i_k=2,\end{cases} \tag{21}$$

and for all $i(k)\in\mathcal{I}, k\geq 1$

$$\eta_{i(k)}=\begin{cases} c_{i_1+i_2+\ldots+i_k}, & \text{if } i_k=1;\\ c'_{i_1+i_2+\ldots+i_k}, & \text{if } i_k=2,\end{cases} \tag{22}$$

where the $a_i, b_i, c_i$, $i\geq 1$, are coeficients of (1), the $a'_{i+2}, b'_{i+1}, c'_i$, $i\geq 2$, are obtained from (15).

**Theorem 1.** *Let $\{A_n\}$, $\{B_n\}$ be sequences of complex numbers such that*

$$A_{-1}=1,\ A_0=0,\ A_1=b_1,\ B_{-1}=0,\ B_0=1,\ B_1=c_1,$$

*and*

$$A_n=c_nA_{n-1}+b_nA_{n-2}+a_nA_{n-3},\ n=2,3\ldots,$$

$$B_n=c_nB_{n-1}+b_nB_{n-2}+a_nB_{n-3},\ n=2,3\ldots,$$

*where the $a_n, b_n, c_n$, $n\geq 1$, are complex constants. Then the $A_n$, $B_n$ are the canonical numerator and the canonical denominator of $n$th $\mathcal{B}$-figured approximant of the BCF (2), i.e. $\widehat{f}_n=A_n/B_n$.*

*Proof.* Applying the equality $A_n/B_n = w_1^{(n)}$, $n \geq 1$, we use the recurrent relations (17)–(19) and step by step write the value $A_n/B_n$ in a form of a finite BCF that is equal $\widehat{f}_n$. On the first step we have

$$\frac{A_n}{B_n} = \frac{b_1}{c_1 + w_2^{(n)}} + \frac{a_2}{c_2' + v_3^{(n)}} = \frac{\xi_1}{\eta_1 + w_2^{(n)}} + \frac{\xi_2}{\eta_2 + v_3^{(n)}}.$$

After the second step we get

$$\frac{A_n}{B_n} = \cfrac{\xi_1}{\eta_1 + \cfrac{\xi_{1,1}}{\eta_{1,1} + w_3^{(n)}} + \cfrac{\xi_{1,2}}{\eta_{1,2} + v_4^{(n)}}} + \frac{\xi_2}{\eta_2 + v_3^{(n)}},$$

and after the third step we obtain

$$\frac{A_n}{B_n} = \cfrac{\xi_1}{\eta_1 + \cfrac{\xi_{1,1}}{\eta_{1,1} + \cfrac{\xi_{1,1,1}}{\eta_{1,1,1} + w_4^{(n)}} + \cfrac{\xi_{1,1,2}}{\eta_{1,1,2} + v_5^{(n)}}} + \cfrac{\xi_{1,2}}{\eta_{1,2} + v_4^{(n)}}} + \cfrac{\xi_2}{\eta_2 + \cfrac{\xi_{2,1}}{\eta_{2,1} + w_4^{(n)}} + \cfrac{\xi_{2,2}}{\eta_{2,2} + v_5^{(n)}}},$$

etc. Using the method of mathematical induction we prove that after $m$ steps, $1 < m < n$, we get

$$\frac{A_n}{B_n} = \mathop{\mathrm{D}}_{k=1}^{m} \sum_{i_k=1}^{2} \frac{\xi^*_{i(k)}}{\eta^*_{i(k)}}, \tag{23}$$

where $\xi^*_{i(k)} = \xi_{i(k)}$, if $i_1 + i_2 + \ldots + i_k \leq m$ or $i_1 + i_2 + \ldots + i_k = m + 1$ and $i_k = 2$; if $i_1 + i_2 + \ldots + i_k \leq m - 1$, then $\eta^*_{i(k)} = \eta_{i(k)}$; if $i_k = 1$ and $i_1 + i_2 + \ldots + i_k = m$, then $\eta^*_{i(k)} = \eta_{i(k)} + w_{m+1}^{(n)}$; if $i_k = 2$ and $i_1 + i_2 + \ldots + i_k = m$, then $\eta^*_{i(k)} = \eta_{i(k)} + v_{m+1}^{(n)}$; if $i_k = 2$ and $i_1 + i_2 + \ldots + i_k = m + 1$, then $\eta^*_{i(k)} = \eta_{i(k)} + v_{m+2}^{(n)}$. In all other cases $\dfrac{\xi^*_{i(k)}}{\eta^*_{i(k)}} \equiv \dfrac{0}{1}$.

Let us make the next, $m + 1$, step. Let $i_1 + i_2 + \ldots + i_k = m$, $i_k = 1$, then

$$\eta^*_{i(k)} = \eta_{i(k)} + w_{m+1}^{(n)} = \eta_{i(k)} + \frac{b_{m+1}}{c_{m+1} + w_{m+2}^{(n)}} + \frac{a_{m+2}}{c'_{m+2} + v_{m+3}^{(n)}}$$

or by using (21), (22) we obtain

$$\eta^*_{i(k)} = \eta_{i(k)} + \frac{\xi_{i(k),1}}{\eta_{i(k),1} + w_{m+2}^{(n)}} + \frac{\xi_{i(k),2}}{\eta_{i(k),2} + v_{m+3}^{(n)}}.$$

If $i_1 + i_2 + \ldots + i_k = m$, $i_k = 2$, then

$$\begin{aligned} \eta^*_{i(k)} = \eta_{i(k)} + v_{m+1}^{(n)} &= \eta_{i(k)} + \frac{b'_{m+1}}{c_{m+1} + w_{m+2}^{(n)}} + \frac{a'_{m+2}}{c'_{m+2} + v_{m+3}^{(n)}} \\ &= \eta_{i(k)} + \frac{\xi_{i(k),1}}{\eta_{i(k),1} + w_{m+2}^{(n)}} + \frac{\xi_{i(k),2}}{\eta_{i(k),2} + v_{m+3}^{(n)}}. \end{aligned}$$

If $i_1 + i_2 + \ldots + i_k = m+1$, $i_k = 2$, then $\eta^*_{i(k)} = \eta_{i(k)} + v^{(n)}_{m+2}$.

Hense, we get the equality (23), where $m$ is replaced by $m+1$.

Put $m = n-1$. Then, using the equalities (19) we obtain that $\eta^*_{i(k)} = \eta_{i(k)} + \dfrac{\xi_{i(k),1}}{\eta_{i(k),1}}$ if $i_1 + i_2 + \ldots + i_k = n-1$, and $\eta^*_{i(k)} = \eta_{i(k)}$ if $i_1 + i_2 + \ldots + i_k = n$, $i_k = 2$.

Thus,

$$\frac{A_n}{B_n} = \mathop{\mathrm{D}}_{k=1}^{n} \sum_{i_k=1}^{2} \frac{\xi^*_{i(k)}}{\eta^*_{i(k)}} = \widehat{f}_n.$$

□

**Remark 1.** *A BCF with two branches of branching with arbitrary complex elements*

$$\mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{2} \frac{\alpha_{i(k)}}{\beta_{i(k)}} \tag{24}$$

*can not be transformed into the form* (2), *where the* $\xi_{i(k)}$, $\eta_{i(k)}$, $i(k) \in \mathcal{I}$, *are determined by formulas* (20)–(22). *For calculating canonical numerators and canonical denominators of a BCF with* $N$ *branches of branching,* $N > 1$, *the linear recurrent relations do not hold.*

Let us consider the $n$th $\mathcal{B}$-figured approximants of BCF (24) and (2). Let $n = 2$, then we get second $\mathcal{B}$-figured approximant of the BCF (24) $\widehat{g}_2 = \alpha_1 / \left(\beta_1 + \alpha_{1,1}\beta^{-1}_{1,1}\right) + \alpha_2/\beta_2$, and by using the formulas (20)–(22) and (15) we obtain second $\mathcal{B}$-figured approximant of the BCF (2) $\widehat{f}_2 = b_1 / \left(c_1 + b_2 c_2^{-1}\right) + a_2 / \left(c_1 c_2 + b_2\right)$. If we put $b_1 = \alpha_1$, $c_1 = \beta_1$, $a_2 = \alpha_2$, $b_2 = \alpha_{11}$, $c_2 = \beta_{12}$, then we get that the relation $\beta_1\beta_{12} + \alpha_{11} = \beta_2$ must hold. But the $\beta_2$ is arbitrary. Hense, this is the case that illustrates the truth of the Remark 1.

**Theorem 2.** *Let the coefficients* $a_n$, $b_n$, $c_n$, $n \geq 2$, *of equation* (1) *be positive real numbers such that*

$$\sum_{k=2}^{\infty} \mu_k = \infty, \tag{25}$$

*where*

$$\begin{aligned}
\mu_k &= \min_{k \leq j \leq 2k} \left\{ \frac{M_j}{R_{j+1}}, \frac{M_{j+1}}{R'_{j+2}} \right\}, \quad k \geq 2, \\
M_j &= c_j c'_j c_{j+1} c'_{j+2}, \quad j \geq 2, \\
R_j &= b_j c'_{j-1} c'_{j+1} + a_{j+1} c_{j-1} c_j, \quad j \geq 3, \\
R'_j &= b'_j c'_{j-1} c'_{j+1} + a'_{j+1} c_{j-1} c_j, \quad j \geq 4,
\end{aligned}$$

*and* $a'_{i+2}$, $b'_{i+1}$, $c'_i$, $i \geq 2$, *are determined by* (15). *Then the BCF* (2), *whose elements satisfy relations* (20)–(22), *$\mathcal{B}$-figured converges.*

*Proof.* Let us show that the elements of the BFC (2) satisfy the conditions of the Theorem 3.11 [7, p. 85]. For this, we consider the following expressions $d_{i(k+1)} = \eta_{i(k)}\eta_{i(k+1)}/\xi_{i(k+1)}$, $i(k+1) \in \mathcal{I}$, $k \geq 2$. If we fix $i(k-1) \in \mathcal{I}$, $k \geq 2$, using the relations (21), (22), we obtain

$$d_{i(k-1),1,1} = \frac{c_j c_{j+1}}{b_{j+1}}, \; d_{i(k-1),2,1} = \frac{c_{j+1} c_{j+2}}{b'_{j+2}}, \; d_{i(k-1),1,2} = \frac{c'_j c'_{j+2}}{a_{j+2}}, \; d_{i(k-1),2,2} = \frac{c'_{j+1} c'_{j+3}}{a'_{j+3}},$$

where $j = \sum_{l=1}^{k-1} i_l + 1$. From this we obtain

$$\min_{i(k+1)\in\mathcal{I},\, k\geq 2} \left\{ d_{i(k+1)} \right\} = \min_{k\leq j\leq 2k,\, k\geq 2} \left\{ \frac{c_j c_{j+1}}{b_{j+1}}, \frac{c_{j+1} c_{j+2}}{b'_{j+2}}, \frac{c'_j c'_{j+2}}{a_{j+2}}, \frac{c'_{j+1} c'_{j+3}}{a'_{j+3}} \right\}$$
$$\geq \min_{k\leq j\leq 2k,\, k\geq 2} \left\{ \frac{M_j}{R_{j+1}}, \frac{M'_{j+1}}{R'_{j+2}} \right\} = \mu_k.$$

Now from (25) it follows that the elements of the BFC (2) satisfy the conditions of the Theorem 3.11 [7, p. 85]. This means that the BFC (2) converges.

Finally, by the Theorem 2.2 [7, p. 48], the BFC (2) $\mathcal{B}$-figured converges. □

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Біланик І.Б., Боднар Д.І., Буяк Л.М. *Зображення відношення розв'язків чотиричленного лінійного рекурентного співвідношення у вигляді гіллястого ланцюгового дробу* // Карпатські матем. публ. — 2019. — Т.11, №1. — C. 33–41.

Відношення двох лінійно незалежних розв'язків чотиричленного лінійного рекурентного співвідношення за аналогією з неперервними дробами представлено у вигляду гіллястого ланцюгового дробу з двома гілками розгалуження. Знайдено формули частинних чисельників та частинних знаменників цього гіллястого ланцюгового дробу. Розв'язки різницевого рівняння є канонічними чисельниками і канонічними знаменниками $\mathcal{B}$-фігурних підхідних дробів. Часто використовують два типи фігурних підхідних дробів: $\mathcal{A}$-фігурні і $\mathcal{B}$-фігурні. $n$–ий $\mathcal{A}$-фігурний підхідний дріб гіллястого ланцюгового дробу отримується додаванням наступної частинної частки до $(n-1)$–го $\mathcal{A}$-фігурного підхідного дробу. $n$–ий $\mathcal{B}$-фігурний підхідний дріб гіллястого ланцюгового дробу є гіллястий ланцюговий дріб, що є його частиною і містить всі ті елементи, сума індексів яких менша, або рівна $n$. $\mathcal{A}$-фігурні підхідні дроби використовуються при доведенні формул для канонічних чисельників і знаменників у вигляді визначників, $\mathcal{B}$-фігурні підхідні дроби – у задачах відповідності між кратними степеневими рядами і гіллястими ланцюговими дробами. Загальний гіллястий ланцюговий дріб не можна звести до побудованого гіллястого ланцюгового дробу. Для обчислення канонічних чисельників і канонічних знаменників гіллястих ланцюгових дробів з $N$, $N > 1$, гілками розгалуження не справджуються лінійні рекурентні співвідношення. Досліджена $\mathcal{B}$-фігурна збіжність побудованого дробу у випадку, коли коефіцієнтами рекурентного співвідношення є дійсні додатні числа.

*Ключові слова і фрази:* гіллястий ланцюговий дріб, рекурентне співвідношення.

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