DOI 10.1007/s10958-020-04729-w

Journal of Mathematical Sciences, Vol. 246, No. 2, April, 2020

ON THE CONVERGENCE OF BRANCHED CONTINUED FRACTIONS OF A SPECIAL FORM IN ANGULAR DOMAINS

D. I. Bodnar[1] **and I. B. Bilanyk**[2]

UDC 517.524

We study the angular domains of convergence of branched continued fractions of a special form with complex partial denominators. By using the sufficient criterion of convergence of these fractions with positive elements and the Stieltjes–Vitali theorem, we establish a many-dimensional analog of the van Vleck convergence criterion for continued fractions, as well as some other sufficient criteria of convergence under certain additional conditions imposed on the elements of fractions. The estimates of the rate of convergence for branched continued fractions of special form are obtained in these angular domains.

In the analytic theory of continued fractions, significant attention is given to the problem of convergence. Numerous criteria of convergence are formulated as the theorems on the sets of convergence. Thus, the researchers consider circular, oval, parabolic, angular, and other domains of convergence. As one of the classical criteria of convergence of continued fractions with partial numerators equal to one is the van Vleck criterion of convergence [7, 15–17] corresponding to the convergence of continued fractions just in the angular domains.

This theorem was generalized in [13], where various estimates for the rate of convergence of the convergents were obtained. In particular, the following assertion was proved:

Theorem 1. *Assume that the elements of a continued fraction*

$$b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \frac{1}{b_i}$$

satisfy the conditions

$$b_i \neq 0, \qquad |\arg b_i| < \theta, \qquad \theta < \frac{\pi}{2}, \qquad i = 0,1,2,\ldots.$$

Then

(1°) *there exist the finite limits of even and odd convergents;*

(2°) *the sequence of convergents* $\{f_n\}$ *converges iff the series* $\sum_{i=1}^{\infty} |b_i|$ *diverges;*

(3°) *the following estimate is true:*

$$|f_m - f_{n-1}| \le \frac{1}{d_n}, \qquad m \ge n,$$

[1] Ternopil National Economic University, Ternopil, Ukraine.

[2] Pidstryhach Institute for Applied Problems in Mechanics and Mathematics, Ukrainian National Academy of Sciences, Lviv, Ukraine.

Translated from Matematychni Metody ta Fizyko-Mekhanichni Polya, Vol. 60, No. 3, pp. 60–69, July–September, 2017. Original article submitted February 8, 2017.

 1072-3374/20/2462–0188

where

$$d_n \geq \frac{\Re(b_1)}{2+\Re(b_1)}\cos\theta \ln\left(1+(\Re(b_1))^2 \min\left\{1, \frac{1}{|b_1|^2}\right\}\cos\theta \sum_{i=1}^{n}|b_i|\right), \qquad n \geq 1. \tag{1}$$

A many-dimensional analog of the van Vleck theorem for branched continued fractions (BCF) with N branches was proved in [4]. A two-dimensional generalization of this theorem under other conditions imposed on the divergence of series formed by elements of a two-dimensional continued fraction was proposed in work [6]. The estimate of the rate of convergence of these fractions in the angular domain was established in [10].

In what follows, we study BCF of a special form

$$b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{1}{b_{i(k)}}, \tag{2}$$

where $b_0, b_{i(k)}$ are complex numbers, $i(k) \in I$,

$$I = \{i(k) = i_1 i_2 \ldots i_k : 1 \leq i_k \leq i_{k-1} \leq \ldots \leq i_0,\ k \geq 1,\ i_0 = N\},$$

and the value of i_0 determines the dimension of the fraction. For fractions of this kind, we establish a many-dimensional generalization of the van Vleck theorem and estimate the rates of convergence in angular domains.

The BCF of a special form were studied in numerous works including, in particular [1, 2, 3, 5, 8, 9, 11, 12, 14], where the authors established analogs of the criteria of convergence of continued fractions proposed by Worpitzky, Pringsheim, and Leighton and Wall, parabolic theorems, etc.

In this case, the researchers extensively used the sufficient criterion of convergence of the BCF of special form with positive elements proposed in [11]:

Theorem 2. *The BCF (2) with positive elements converges if the series*

$$\sum_{p=1}^{\infty} b_{m[p]}, \qquad m = 1, \ldots, N,$$

$$\sum_{p=1}^{\infty} b_{i(n)m[p]}, \qquad m = 1, \ldots, N, \qquad i(n) \in I^{(m+1)},$$

where

$$I^{(m)} = \{i(n) = i_1 i_2 \ldots i_n : m \leq i_n \leq i_{n-1} \leq \ldots \leq i_0,\ n \geq 1,\ i_0 = N\}, \tag{3}$$

$$m = 2, \ldots, N, \qquad m[p] = \underbrace{mm\ldots m}_{p},$$

are divergent.

By using the technique of sets of elements and sets of values, the Stieltjes–Vitali theorem [7, Theorem 4.30], and a method proposed in [6], we prove the following many-dimensional analog of the van Vleck theorem:

Theorem 3. *Assume that the partial denominators of the BCF (2) belong to the domain*

$$G(\varepsilon) = \left\{ z \in \mathbb{C} : z \neq 0, |\arg z| < \frac{\pi}{2} - \varepsilon \right\}, \tag{4}$$

where ε *is any positive number,* $0 < \varepsilon < \frac{\pi}{2}$.

Then

(1°) *every* n*th approximation* f_n *of the BCF (2) belongs to domain (4);*

(2°) *there exist finite limits of even and odd convergents;*

(3°) *the BCF (2) converges if the series*

$$\begin{gathered} \sum_{p=1}^{\infty} |b_{m[p]}|, \qquad m = 1,\ldots,N, \\ \sum_{p=1}^{\infty} |b_{i(n)m[p]}|, \qquad m = 1,\ldots,N-1, \qquad i(n) \in I^{(m+1)}, \end{gathered} \tag{5}$$

are divergent.

Proof. In view of the convexity and symmetry of domain (4) and the properties of a linear-fractional mapping, it is easy to set that $f_n \in G(\varepsilon)$ $\forall n \in \mathbb{N}$.

We now consider a functional BCF

$$\mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{1}{b_{i(k)}(z)}, \tag{6}$$

where

$$b_{i(k)}(z) = |b_{i(k)}| e^{i \arg b_{i(k)} z}.$$

If $z \in D(\varepsilon)$, where

$$D(\varepsilon) = \left\{ z \in \mathbb{C} : |\Re(z)| < 1 + \frac{\varepsilon}{\pi - 2\varepsilon}, |\Im(z)| < 1 \right\},$$

then

$$b_{i(k)}(z) \in G\left(\frac{\varepsilon}{2}\right).$$

For a sequence of holomorphic functions $\{f_n(z)\}$, where $f_n(z)$ is the nth convergent of the BCF (6), $n \geq 1$, the conditions of the Stieltjes–Vitali theorem with $\Delta = \{z \in \mathbb{C} : |\Re(z)| = 0, |\Im(z)| < 1\}$, are satisfied in the domain $D(\varepsilon)$.

Let $z \in \Delta$. Then the BCF (6) can be represented in the form

$$\mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{1}{\tilde{b}_{i(k)}}, \tag{7}$$

where

$$\tilde{b}_{i(k)} = |b_{i(k)}| e^{-\arg b_{i(k)} \Im(z)}.$$

The "fork" property for the BCF of special form with positive elements (7) implies that its even and odd convergents have limits. Therefore, by the Stieltjes–Vitali theorem, there exist finite limits of even and odd convergents of the BCF (2).

The fact that series (5) are divergent implies that the series

$$\sum_{p=1}^{\infty} \tilde{b}_{m[p]}, \quad m = 1, \ldots, N, \qquad \text{and} \qquad \sum_{p=1}^{\infty} \tilde{b}_{i(n)m[p]}, \quad m = 1, \ldots, N-1, \ i(n) \in I^{(m+1)},$$

are also divergent. It follows from Theorem 2 that the BCF (7) is convergent. This means that the sequence of holomorphic functions $\{f_n(z)\}$ converges for $z \in \Delta$. It is clear that $\Delta \subset D(\varepsilon)$. Hence, by using the Stieltjes–Vitali theorem with, e.g., $a = -1$ and $b = -2$, we can conclude that the BCF (6) converges on any compact set $D(\varepsilon)$ and, in particular, on the set $\{1\}$, i.e., for $z = 1$. Thus, the BCF (2) is convergent.

The theorem is proved.

The next theorem enables us to estimate the rate of convergence of the BCF of special form with partial numerators equal to one and partial denominators that are complex numbers lying in a certain angular domain.

Theorem 4. *Suppose that the elements of the BCF (2) satisfy the conditions*

$$\Re(b_{i(k)}) > 0, \qquad |\arg b_{i(k)}| < \theta, \qquad \theta < \frac{\pi}{2}, \qquad i(k) \in I, \tag{8}$$

and the infinite product

$$\prod_{k=1}^{\infty} \left(\frac{1}{\cos\theta} (1 - \nu_k) \right),$$

where

$$\nu_k = \min \left\{ \frac{\Re(b_{i(k)})}{|b_{i(k)}| + \sum_{i_{k+1}=1}^{i_k} \frac{1}{\Re(b_{i(k+1)})}} : i(k) \in I_k \right\}, \tag{9}$$

$$I_k = \{i(k) = i_1 i_2 \ldots i_k : 1 \le i_k \le i_{k-1} \le \ldots \le i_0,\ i_0 = N\}, \qquad k = 1,2,3,\ldots,$$

diverges to zero. Then the BCF (2) converges, and the following estimate of the rate of convergence is true:

$$|f_n - f_p| \le \frac{2N}{\min(\Re(b_{i_1}) : i_1 = 1,2,\ldots,N)} \left(\frac{1}{\cos\theta}\right)^{2s} \prod_{k=1}^{2s} (1 - \nu_k),$$
$$n > p, \qquad s = \left[\frac{p}{2}\right]. \tag{10}$$

Proof. In view of condition (8) and the properties of linear-fractional mappings, it is easy to see that, for the remainders of the nth convergents of BCF (2)

$$Q_{i(n)}^{(n)} = b_{i(n)}, \qquad n \ge 1,$$

$$Q_{i(k)}^{(n)} = b_{i(k)} + \sum_{i_{k+1}=1}^{i_k} \frac{1}{Q_{i(k+1)}^{(n)}}, \qquad k = 1,2,\ldots,n-1, \qquad n \ge 2,$$

the following estimate is true:

$$\left|\arg Q_{i(k)}^{(n)}\right| < \theta, \qquad k = 1,2,\ldots,n, \qquad n = 1,2,\ldots.$$

Hence,

$$\left|Q_{i(k)}^{(n)}\right| \ge \Re(Q_{i(k)}^{(n)}) \ge \Re(b_{i(k)}) + \sum_{i_{k+1}=1}^{i_k} \frac{\cos\theta}{\left|Q_{i(k+1)}^{(n)}\right|}, \qquad k = 1,2,\ldots,n-1, \qquad n \ge 2. \tag{11}$$

For the absolute value of the difference of convergents of the BCF (2) with $n > 2m$, we arrive at the estimate [3]

$$|f_n - f_{2m}| \le \sum_{i_1=1}^{N} \frac{1}{\left|Q_{i(1)}^{(n)}\right|} \sum_{i_2=1}^{i_1} \ldots \sum_{i_{2m}=1}^{i_{2m-1}} \frac{1}{\prod_{k=1}^{m-1}\left|Q_{i(2k)}^{(n)} Q_{i(2k+1)}^{(n)}\right| \prod_{k=1}^{m}\left|Q_{i(2k-1)}^{(2m)} Q_{i(2k)}^{(2m)}\right|}$$
$$\times \sum_{i_{2m+1}=1}^{i_{2m}} \frac{1}{\left|Q_{i(2m)}^{(n)} Q_{i(2m+1)}^{(n)}\right|}.$$

By using inequalities (11) and the fact that the function

$$y = \frac{x}{b+x}, \qquad b \ge 0,$$

monotonically increases, for positive values of the variable x, we arrive at the estimate

$$\sum_{i_{2m+1}=1}^{i_{2m}} \frac{1}{\left|Q_{i(2m)}^{(n)} Q_{i(2m+1)}^{(n)}\right|} \leq \frac{1}{\cos\theta} \frac{\sum_{i_{2m+1}=1}^{i_{2m}} \frac{\cos\theta}{\left|Q_{i(2m+1)}^{(n)}\right|}}{\Re(b_{i(2m)}) + \sum_{i_{2m+1}=1}^{i_{2m}} \frac{\cos\theta}{\left|Q_{i(2m+1)}^{(n)}\right|}}$$

$$\leq \frac{1}{\cos\theta} \frac{\left|Q_{i(2m)}^{(n)}\right| - \Re(b_{i(2m)})}{\left|Q_{i(2m)}^{(n)}\right|}$$

$$\leq \frac{1}{\cos\theta} \left(1 - \frac{\Re(b_{i(2m)})}{\left|b_{i(2m)}\right| + \sum_{i_{2m+1}=1}^{i_{2m}} \frac{1}{\Re(b_{i(2m+1)})}} \right)$$

$$\leq \frac{1}{\cos\theta}(1 - \nu_{2m}),$$

where ν_{2m} is defined according to (9).

We now perform similar analysis for the sums

$$\sum_{i_k=1}^{i_{k-1}} \frac{1}{\left|Q_{i(k-1)}^{(s)} Q_{i(k)}^{(s)}\right|}, \quad k = 2m, \quad 2m-1, \ldots, 2,$$

where $s = 2m$ if k is even and $s = n$ if k is odd. In view of the relation

$$\sum_{i_1=1}^{N} \frac{1}{\left|Q_{i(1)}^{(n)}\right|} \leq \frac{2N}{\min\limits_{i_1} \Re(b_{i_1})},$$

we get:

$$|f_n - f_{2m}| \leq \frac{N}{\min\limits_{i_1} \Re(b_{i_1})} \left(\frac{1}{\cos\theta} \right)^{2m} \prod_{k=1}^{2m} (1 - \nu_k), \qquad n > 2m.$$

In view of the triangle inequality

$$\left|f_n - f_p\right| \leq \left|f_n - f_{2s}\right| + \left|f_p - f_{2s}\right|, \; n \leq p, \; s = \left[\frac{p}{2}\right],$$

this yields estimate (10). The conditions of the theorem imply that $\left|f_n - f_p\right| \to 0$ as $p \to \infty$.

The theorem is proved.

The following assertion establishes an estimate for the rate of convergence of the BCF of special form in a domain, which is a subset of the angular domain (8).

Theorem 5. *Assume that the elements of an N-dimensional BCF (2) of special form satisfy the conditions*

$$\Re(b_{i(k)}) > \delta, \qquad 0<\delta<1, \qquad \left|\arg b_{i(k)}\right| < \theta, \qquad \theta < \frac{\pi}{4}, \qquad i(k)\in I. \tag{12}$$

Then the BCF (2) converges and the rate of convergence can be estimated as follows:

$$|f_m - f_{Nn}| < \frac{M_N}{\ln(1+\alpha n)}, \qquad m \ge Nn,$$

where M_N and α are positive constants independent of n and m.

Proof. For $N=1$, in view of Theorem 1 and, in particular, relations (1), the convergents f_m of the continued fractions whose elements satisfy conditions (12) can be estimated as follows:

$$d_n \ge \frac{\delta\cos\theta}{2+\delta}\ln(1+\alpha n), \qquad n \ge 1,$$

where

$$\alpha = \min\{\delta^3\cos\theta, \delta\cos^3\theta\}.$$

Thus,

$$|f_m - f_n| \le \frac{1}{d_{n+1}} < \frac{1}{d_n} < \frac{2+\delta}{\delta\cos\theta}\frac{1}{\ln(1+\alpha n)}, \qquad m \ge n. \tag{13}$$

The proof of the theorem is performed by induction on the dimension N of the BCF.

Let $N=2$ and let

$$f_n = b_0^{(1,n)} + \mathop{\mathrm{D}}\limits_{k=1}^{n}\frac{1}{b_{2[k]}^{(1,n-k)}}, \qquad n \ge 1,$$

be the nth convergent of the BCF (2) in which $i_0 = 2$ and, in view of notation (3),

$$b_0^{(1,n)} = b_0 + \mathop{\mathrm{D}}\limits_{\ell=1}^{n}\frac{1}{b_{1[\ell]}}, \qquad b_{2[k]}^{(1,0)} = b_{2[k]}, \qquad b_{2[k]}^{(1,s)} = b_{2[k]} + \mathop{\mathrm{D}}\limits_{\ell=1}^{s}\frac{1}{b_{2[k]2[\ell]}},$$

$$k = 1,2,\ldots,n, \qquad s = 1,2,\ldots,n-1.$$

Let

$$\tilde{f}_n = b_0^{(1)} + \mathop{\mathrm{D}}\limits_{k=1}^{n}\frac{1}{b_{2[k]}^{(1)}}$$

be the nth convergent of the continued fraction obtained as a result of the convolution of all continued fractions in the BCF (2). In other words, we get

$$b_0^{(1)} = b_0 + \mathop{\mathrm{D}}_{\ell=1}^{\infty} \frac{1}{b_{1[\ell]}}, \qquad b_{2[k]}^{(1)} = b_{2[k]} + \mathop{\mathrm{D}}_{\ell=1}^{\infty} \frac{1}{b_{2[k]1[\ell]}}, \qquad k = 1, 2, \ldots, n.$$

These continued fractions are convergent by Theorem 3.

We now estimate the quantities $\left|f_p - \tilde{f}_{2n}\right|$, $p \ge 2n$. Let

$$h_p = b_0^{(1)} + \frac{1}{b_{2[1]}^{(1)}} + \ldots + \frac{1}{b_{2[n]}^{(1)}} + \frac{1}{b_{2[n+1]}^{(1,p-n-1)}} + \ldots + \frac{1}{b_{2[p]}^{(1,0)}}.$$

Then the triangle inequality implies that

$$\left|f_p - \tilde{f}_{2n}\right| \le \left|f_p - h_p\right| + \left|h_p - \tilde{f}_{2n}\right|. \tag{14}$$

We find

$$\left|f_p - h_p\right| \le \left|b_0^{(1,p)} - b_0^{(1)}\right| + \sum_{k=1}^{n} \frac{\left|b_{2[k]}^{(1,p-k)} - b_{2[k]}^{(1)}\right|}{\prod_{s=1}^{k}\left|\tilde{Q}_{2[s]}^{(p)} Q_{2[s]}^{(p)}\right|},$$

where $\tilde{Q}_{2[s]}^{(p)}$ is the sth remainder of the fraction h_p.

We now estimate the products in the denominators of this relation. If $k = 2\ell$, then

$$\prod_{s=1}^{2\ell}\left|\tilde{Q}_{2[s]}^{(p)} Q_{2[s]}^{(p)}\right| = \prod_{s=1}^{\ell}\left|\tilde{Q}_{2[2s-1]}^{(p)} \tilde{Q}_{2[2s]}^{(p)}\right| \left|Q_{2[2s-1]}^{(p)} Q_{2[2s]}^{(p)}\right|$$

$$\ge \prod_{s=1}^{\ell} (\Re(b_{2[2s]} b_{2[2s-1]}) + 1)^2 \ge (\delta^2 \cos 2\theta + 1)^{2\ell}.$$

At the same time, if $k = 2\ell + 1$, then

$$\prod_{s=1}^{2\ell+1}\left|\tilde{Q}_{2[s]}^{(p)} Q_{2[s]}^{(p)}\right| = \left|\tilde{Q}_{2[1]}^{(p)} Q_{2[1]}^{(p)}\right| \prod_{s=1}^{\ell}\left|\tilde{Q}_{2[2s]}^{(p)} \tilde{Q}_{2[2s+1]}^{(p)}\right| \left|Q_{2[2s]}^{(p)} Q_{2[2s+1]}^{(p)}\right|$$

$$\ge (\Re(b_{2[1]}))^2 \prod_{s=1}^{\ell} (\Re(b_{2[2s]} b_{2[2s+1]}) + 1)^2 \ge \delta^2 (\delta^2 \cos 2\theta + 1)^{2\ell}.$$

Hence,

$$\prod_{s=1}^{k}\left|\tilde{Q}_{2[s]}^{(p)} Q_{2[s]}^{(p)}\right| \ge \delta^{1-(-1)^k} (\delta^2 \cos 2\theta + 1)^{2\left[\frac{k}{2}\right]}.$$

The quantity $\left|b_{2[k]}^{(1,p-k)} - b_{2[k]}^{(1)}\right|$ can be estimated by using Theorem 1 and, in particular, inequality (13) obtained as a consequence of this theorem:

$$\left|b_{2[k]}^{(1,p-k)} - b_{2[k]}^{(1)}\right| < \frac{2+\delta}{\delta\cos\theta}\frac{1}{\ln(1+\alpha n)}, \qquad k = 1,\ldots,n, \qquad p \ge 2n.$$

Thus,

$$\left|f_p - h_p\right| < \frac{2+\delta}{\delta\cos\theta}\frac{1}{\ln(1+\alpha n)}\left(1 + \frac{1}{\delta^2} + \frac{2}{\delta^2}\sum_{k=1}^{\infty}(\delta^2\cos 2\theta + 1)^{-2k}\right)$$

$$\le \frac{(2+\delta)A}{\delta\cos\theta}\frac{1}{\ln(1+\alpha n)},$$

where

$$A = 1 + \frac{1}{\delta^2} + \frac{2}{\delta^4\cos 2\theta(\delta^2\cos 2\theta + 2)}.$$

We now estimate the second term on the right-hand side of inequality (14) by using Theorem 1:

$$\left|h_p - \tilde{f}_{2n}\right| \le \left|h_p - \tilde{f}_n\right| + \left|\tilde{f}_{2n} - \tilde{f}_n\right| < \frac{2}{D_n},$$

where

$$D_n \ge \frac{\Re(b_0^{(1)})\cos\theta}{2+\Re(b_0^{(1)})}\ln\left(1 + (\Re(b_0^{(1)}))^2\min\left\{1, \frac{1}{\left|b_0^{(1)}\right|^2}\right\}\cos\theta\sum_{s=1}^{n}\left|b_{2[s]}^{(1)}\right|\right).$$

Estimating the last quantity, we get

$$D_n \ge \frac{\delta\cos\theta}{2+\delta}\ln(1+\alpha n).$$

Thus,

$$\left|h_p - \tilde{f}_{2n}\right| < \frac{2(2+\delta)}{\delta\cos\theta}\frac{1}{\ln(1+\alpha n)}.$$

By using the estimates presented above, we find

$$\left|f_p - \tilde{f}_{2n}\right| < \frac{2+\delta}{\delta\cos\theta}(A+2)\frac{1}{\ln(1+\alpha n)}, \qquad p \ge 2n.$$

Hence,

$$\begin{aligned}|f_m - f_{2n}| &\le |f_m - \tilde{f}_{2n}| + |f_{2n} - \tilde{f}_{2n}| \\ &< \frac{2+\delta}{\delta\cos\theta} 2(A+2) \frac{1}{\ln(1+\alpha n)} \\ &= \frac{2+\delta}{\delta\cos\theta} K_2 \frac{1}{\ln(1+\alpha n)} = \frac{M_2}{\ln(1+\alpha n)}, \qquad m \ge 2n,\end{aligned}$$

where

$$M_2 = \frac{2+\delta}{\delta\cos\theta} K_2, \qquad K_2 = 2(A+2).$$

Assume that, all N-dimensional BCF of special form (2) with $i_0 = N$, where $N \le r-1$, whose elements satisfy condition (12) can be estimated as follows:

$$\left| f_m - f_{(r-1)n} \right| < \frac{M_{r-1}}{\ln(1+\alpha n)}, \qquad m \ge (r-1)n, \tag{15}$$

where

$$M_{r-1} = \frac{2+\delta}{\delta\cos\theta} K_{r-1}, \qquad K_{r-1} = 4(K_{r-2}A+1). \tag{16}$$

We now prove the validity of estimate (15) for $N = r$. Consider the r-dimensional BCF of special form (2) with $i_0 = r$. Its nth convergent can be represented in the form

$$f_n = b_0^{(r-1,n)} + \mathop{\mathrm{D}}_{k=1}^{n} \frac{1}{b_{r[k]}^{(r-1,n-k)}}, \qquad n \ge 1,$$

where

$$b_0^{(r-1,n)} = b_0 + \mathop{\mathrm{D}}_{\ell=1}^{n} \sum_{i_\ell=1}^{i_{\ell-1}} \frac{1}{b_{i(\ell)}}, \qquad b_{r[k]}^{(r-1,s)} = b_{r[k]} + \mathop{\mathrm{D}}_{\ell=1}^{s} \sum_{i_\ell=1}^{i_{\ell-1}} \frac{1}{b_{r[k]i(\ell)}}$$

are the sth convergents of all $(r-1)$-dimensional BCF appearing in the BCF (2); $k = 1,2,\ldots,n-1$, $s = 1,2,\ldots,n$, $i_0 = r-1$.

By

$$\hat{f}_n = b_0^{(r-1)} + \mathop{\mathrm{D}}_{k=1}^{n} \frac{1}{b_{r[k]}^{(r-1)}},$$

we denote the nth convergent of the continued fraction formed as a result of the convolution of all $(r-1)$-dimensional BCF in the BCF (2), i.e.,

$$b_0^{(r-1)} = b_0 + \mathop{\mathrm{D}}_{\ell=1}^{n} \sum_{i_\ell=1}^{i_{\ell-1}} \frac{1}{b_{i(\ell)}}, \qquad b_{r[k]}^{(r-1)} = b_{r[k]} + \mathop{\mathrm{D}}_{\ell=1}^{s} \sum_{i_\ell=1}^{i_{\ell-1}} \frac{1}{b_{r[k]i(\ell)}},$$

$$i_0 = r-1, \qquad k = 1, 2, \ldots, n.$$

This convolution can be performed by virtue of Theorem 3.

We now estimate the modulus of the difference $\left| f_p - \hat{f}_{rn} \right|$, $p \ge rn$:

$$\left| f_p - \hat{f}_{rn} \right| \le \left| f_p - g_p \right| + \left| g_p - \hat{f}_{rn} \right|,$$

where

$$g_p = b_0^{(r-1)} + \frac{1}{b_{r[1]}^{(r-1)}} + \ldots + \frac{1}{b_{r[(r-1)n]}^{(r-1)}} + \frac{1}{b_{r[(r-1)n+1]}^{(r-1,p-(r-1)n-1)}} + \ldots + \frac{1}{b_{r[p]}^{(r-1,0)}}.$$

For the first term, we obtain

$$\left| f_p - g_p \right| \le \left| b_0^{(r-1,p)} - b_0^{(r-1)} \right| + \sum_{k=1}^{(r-1)n} \frac{\left| b_{r[k]}^{(r-1,p-k)} - b_{r[k]}^{(r-1)} \right|}{\prod_{s=1}^{k} \left| \hat{Q}_{r[s]}^{(p)} Q_{r[s]}^{(p)} \right|}, \tag{17}$$

where $\hat{Q}_{r[s]}^{(p)}$ is the sth remainder of the fraction g_p.

Estimating the products in the denominators of (17) by using the reasoning similar to the reasoning used for $N = 2$, we get

$$\prod_{s=1}^{k} \left| \hat{Q}_{r[s]}^{(p)} Q_{r[s]}^{(p)} \right| \ge \delta^{1-(-1)^k} \left(\delta^2 \cos 2\theta + 1 \right)^{2\left[\frac{k}{2}\right]}.$$

We now estimate the numerators in (17). It follows from inequality (15) that

$$\left| b_{r[k]}^{(r-1,p-k)} - b_{r[k]}^{(r-1)} \right| \le \left| b_{r[k]}^{(r-1,p-k)} - b_{r[k]}^{(r-1,r-1)} \right| + \left| b_{r[k]}^{(r-1)} - b_{r[k]}^{(r-1,r-1)} \right|$$

$$< \frac{2M_{r-1}}{\ln(1+\alpha n)}, \qquad k = 0, \ldots, (r-1)n, \qquad p \ge rn.$$

Hence,

$$\left| f_p - g_p \right| < \frac{2M_{r-1}}{\ln(1+\alpha n)} A.$$

Estimating the quantity $\left| g_p - \hat{f}_{rn} \right|$, by virtue of Theorem 1, we obtain

$$\left| g_p - \hat{f}_{rn} \right| \le \left| g_p - \hat{f}_{(r-1)n} \right| + \left| \hat{f}_{rn} - \hat{f}_{(r-1)n} \right| < \frac{2(2+\delta)}{\delta \cos\theta} \frac{1}{\ln(1+\alpha n)}.$$

The estimates presented above and relations (16) imply that

$$\left|f_p - \hat{f}_{rn}\right| < \frac{(2+\delta)}{\delta\cos\theta}(2AK_{r-1}+2)\frac{1}{\ln(1+\alpha n)}, \qquad p \ge rn\,.$$

Thus,

$$\begin{aligned}|f_m - f_{rn}| &\le \left|f_m - \hat{f}_{rn}\right| + \left|f_{rn} - \hat{f}_{rn}\right| \\ &< \frac{(2+\delta)}{\delta\cos\theta}4(AK_{r-1}+1)\frac{1}{\ln(1+\alpha n)} \\ &= \frac{(2+\delta)}{\delta\cos\theta}K_r\frac{1}{\ln(1+\alpha n)} = \frac{M_r}{\ln(1+\alpha n)}, \qquad m \ge rn\,,\end{aligned}$$

where

$$M_r = \frac{2+\delta}{\delta\cos}K_r\,, \qquad K_r = 4(AK_{r-1}+1)\,.$$

Hence, the convergents f_k of the N-dimensional BCF (2), where N is an arbitrary fixed natural number, satisfy the estimate

$$|f_m - f_{Nn}| < \frac{M_N}{\ln(1+\alpha n)}, \qquad m \ge Nn\,,$$

where $M_N = \dfrac{2+\delta}{\delta\cos\theta}K_N$, $N \ge 1$, $K_1 = 1$, $K_2 = 2(A+2)$, $K_N = 4(K_{N-1}A+1)$, and $N \ge 3$.

The theorem is proved.

Remark 1. The assertion of the theorem remains valid if conditions (12) are replaced by the conditions

(a) $\Re(b_{i(k)}) > 0$;

(b) $\begin{cases}\Im(b_{i(2\ell)}) > 0, \\ \Im(b_{i(2\ell+1)}) < 0,\end{cases}$ or $\begin{cases}\Im(b_{i(2\ell)}) < 0, \\ \Im(b_{i(2\ell+1)}) > 0,\end{cases}$ $\ell = 1,2,\ldots.$

REFERENCES

1. T. M. Antonova, "The rate of convergence of branched continued fractions of special form," *Volyn. Mat. Visn.*, Issue 6, 5–11 (1999).
2. T. M. Antonova and D. I. Bodnar, "Domains of convergence of branched continued fractions of special form," in: *Theory of Approximation of Functions and Its Applications* [in Ukrainian], Pratsi Inst. Mat. Nats. Akad. Nauk Ukr., **31** (2000), pp. 19–32.
3. O. E. Baran, "Some circular regions of convergence for branched continued fractions of a special form," *Mat. Met. Fiz.-Mekh. Polya*, **56,** No. 3, 7–14 (2013); ***English translation****: J. Math. Sci.*, **205**, No. 4, 491–500 (2015), https://doi.org/10.1007/s10958-015-2262-3.

4. D. I. Bodnar, *Branched Continued Fractions* [in Russian], Naukova Dumka, Kiev (1986).
5. D. I. Bodnar and M. M. Bubnyak, "Estimates of the rate of pointwise and uniform convergence for one-periodic branched continued fractions of a special form," *Mat. Met. Fiz.-Mekh. Polya,* **56,** No. 4, 24–32 (2013); ***English translation****: J. Math. Sci.,* **208**, No. 3, 289–300 (2015), https://doi.org/10.1007/s10958-015-2446-x.
6. D. I. Bodnar and Kh. Yo. Kuchminska, "An analog of the van Vleck theorem for two-dimensional continued fractions," *Mat. Met. Fiz.-Mekh. Polya,* **42,** No. 4, 21–25 (1999).
7. W. B. Jones and W. J. Thron, *Continued Fractions: Analytic Theory and Applications,* Addison-Wesley, Reading, MA (1980).
8. R. I. Dmytryshyn and O. E. Baran, "Some types of branched continued fractions corresponding to multiple power series," in: *Theory of Approximation of Functions and Its Applications* [in Ukrainian], Pratsi Inst. Mat. Nats. Acad. Nauk Ukr., **31** (2000), pp. 82–92.
9. R. I. Dmytryshyn, "Associated branched continued fractions with two independent variables," *Ukr. Mat. Zh.,* **66,** No. 9, 1175–1184 (2014); ***English translation****: Ukr. Math. J.,* **66**, No. 9, 1312–1323 (2015), https://doi.org/10.1007/s11253-015-1011-6.
10. O. M. Sus', "Estimation of the rate of convergence of two-dimensional continued fractions with complex elements," *Prykl. Probl. Mekh. Mat.,* Issue 6, 115–123 (2008).
11. D. I. Bodnar and I. B. Bilanyk, "Convergence criterion for branched continued fractions of the special form with positive elements," *Karpat. Mat. Publ.,* **9**, No. 1, 13–21 (2017), https://doi.org/10.15330/cmp.9.1.13-21.
12. R. I. Dmytryshyn, "On the convergence criterion for branched continued fractions with independent variables," *Karpat. Mat. Publ.,* **9**, No. 2, 120–127 (2017), https://doi.org/10.15330/cmp.9.2.120-127.
13. W. B. Gragg and D. D. Warner, "Two constructive results in continued fractions," *SIAM J. Numer. Anal.,* **20,** No. 6, 1187–1197 (1983), https://doi.org/10.1137/0720088.
14. Kh. Yo. Kuchminska, "A Worpitzky boundary theorem for branched continued fractions of the special form," *Karpat. Mat. Publ.,* **8**, No. 2, 272–278 (2016), https://doi.org/10.15330/cmp.8.2.272-278.
15. L. Lorentzen and H. Waadeland, *Continued Fractions,* Vol. 1: *Convergence Theory,* Atlantis Press/Word Scientific, Amsterdam, Paris (2008).
16. E. B. Van Vleck, "On the convergence of continued fractions with complex elements," *Trans. Amer. Math. Soc.,* **2**, No. 3**,** 215–233 (1901), https://doi.org/10.2307/1986206.
17. H. S. Wall, *Analytic Theory of Continued Fractions,* Van Nostrand, New York (1948).