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Bodnar O.S.[1], Dmytryshyn R.I.[2]

ON THE CONVERGENCE OF MULTIDIMENSIONAL S-FRACTIONS WITH INDEPENDENT VARIABLES

In this paper, we investigate the convergence of multidimensional S-fractions with independent variables, which are a multidimensional generalization of S-fractions. These branched continued fractions are an efficient tool for the approximation of multivariable functions, which are represented by formal multiple power series. For establishing the convergence criteria, we use the convergence continuation theorem to extend the convergence, already known for a small region, to a larger region. As a result, we have shown that the intersection of the interior of the parabola and the open disk is the domain of convergence of a multidimensional S-fraction with independent variables. And, also, we have shown that the interior of the parabola is the domain of convergence of a branched continued fraction, which is reciprocal to the multidimensional S-fraction with independent variables. In addition, we have obtained two new convergence criteria for S-fractions as a consequences from the above mentioned results.

Key words and phrases: convergence, uniform convergence, S-fraction, multidimensional S-fraction with independent variables.

[1] Volodymyr Gnatiuk Ternopil National Pedagogical University, 2 Kryvonosa str., 46027, Ternopil, Ukraine
[2] Vasyl Stefanyk Precarpathian National University, 57 Shevchenka str., 76018, Ivano-Frankivsk, Ukraine
E-mail: bodnarotern@ukr.net (Bodnar O.S.), dmytryshynr@hotmail.com (Dmytryshyn R.I.)

1 Introduction

Establishing convergence criteria for the classes of functional branched continued fractions with independent variables is one of the most important tasks of their studying.

A convergence criteria have been given in [1, 2, 5] for multidimensional regular C-fractions with independent variables

$$1+\sum_{i_1=1}^{N}\frac{a_{i(1)}z_{i_1}}{1}+\sum_{i_2=1}^{i_1}\frac{a_{i(2)}z_{i_2}}{1}+\sum_{i_3=1}^{i_2}\frac{a_{i(3)}z_{i_3}}{1}+\cdots,$$

where the $a_{i(k)}$, $i(k)\in\mathcal{I}_k$, $k\geq 1$, are complex constants such that $a_{i(k)}\neq 0$, $i(k)\in\mathcal{I}_k$, $k\geq 1$,

$$\mathcal{I}_k=\big\{i(k):\ i(k)=(i_1,i_2,\ldots,i_k),\ 1\leq i_p\leq i_{p-1},\ 1\leq p\leq k,\ i_0=N\big\},\quad k\geq 1,$$

denote the sets of multiindices, and where $\mathbf{z}=(z_1,z_2,\ldots,z_N)\in\mathbb{C}^N$, in [8] for multidimensional g-fractions with independent variables

$$\frac{s_0}{1}+\sum_{i_1=1}^{N}\frac{g_{i(1)}z_{i_1}}{1}+\sum_{i_2=1}^{i_1}\frac{g_{i(2)}\big(1-g_{i(1)}\big)z_{i_2}}{1}+\sum_{i_3=1}^{i_2}\frac{g_{i(3)}\big(1-g_{i(2)}\big)z_{i_2}}{1}+\cdots,$$

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where the s_0 is positive constant and the $g_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, are real constants such that $0 < g_{i(k)} < 1$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, and $\mathbf{z} \in \mathbb{C}^N$, in [6] for multidimensional associated fractions with independent variables

$$1 + \sum_{i_1=1}^{N} \frac{b_{i(1)} z_{i_1}}{1} + \sum_{i_2=1}^{i_1} \frac{(-1)^{\delta_{i_1,i_2}} b_{i(2)} z_{i_1} z_{i_2}}{1} + \sum_{i_3=1}^{i_2} \frac{(-1)^{\delta_{i_2,i_3}} b_{i(3)} z_{i_2} z_{i_3}}{1} + \cdots,$$

where the $b_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, are complex constants such that $b_{i(k)} \neq 0$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, and $\delta_{k,p}$ is the Kronecker delta, $1 \leq k, p \leq N$, $\mathbf{z} \in \mathbb{C}^N$, and in [7] for multidimensional J-fractions with independent variables

$$\sum_{i_1=1}^{N} \frac{-p_{i(1)}^2}{q_{i(1)} + z_{i_1}} + \sum_{i_2=1}^{i_1} \frac{-p_{i(2)}^2}{q_{i(2)} + z_{i_2}} + \sum_{i_3=1}^{i_2} \frac{-p_{i(3)}^2}{q_{i(3)} + z_{i_3}} + \cdots,$$

where the $p_{i(k)}$ and $q_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, are complex constants such that $p_{i(k)} \neq 0$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, and $\mathbf{z} \in \mathbb{C}^N$. In this paper, we investigate a convergence of multidimensional S-fraction with independent variables

$$1 + \sum_{i_1=1}^{N} \frac{c_{i(1)} z_{i_1}}{1} + \sum_{i_2=1}^{i_1} \frac{c_{i(2)} z_{i_2}}{1} + \sum_{i_3=1}^{i_2} \frac{c_{i(3)} z_{i_3}}{1} + \cdots, \tag{1}$$

where the $c_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, are real constants such that $c_{i(k)} > 0$, $\mathbf{z} \in \mathbb{C}^N$, and reciprocal to it

$$\frac{1}{1} + \sum_{i_1=1}^{N} \frac{c_{i(1)} z_{i_1}}{1} + \sum_{i_2=1}^{i_1} \frac{c_{i(2)} z_{i_2}}{1} + \sum_{i_3=1}^{i_2} \frac{c_{i(3)} z_{i_3}}{1} + \cdots. \tag{2}$$

We note that the multidimensional S-fraction with independent variables (1) is multidimensional generalization of S-fraction

$$1 + \frac{c_1 z}{1} + \frac{c_2 z}{1} + \frac{c_3 z}{1} + \cdots, \tag{3}$$

where the c_k, $k \geq 1$, are real constants such that $c_k > 0$, $k \geq 1$, $z \in \mathbb{C}$. A convergence result for S-fraction is as follows (see Theorem 4.58 [9, p. 136]).

Theorem 1. *Let (3) be an S-fraction and let $\mathcal{H} = \{z \in \mathbb{C} : \; |\arg(z)| < \pi\}$ be the complex plane cut along the negative real axis. Then the following statements hold.*

(A) The S-fraction (3) converges to a function holomorphic in $\mathcal{H}$ if at least one of the two series $\sum_{k=1}^{\infty} \frac{c_1 c_3 \ldots c_{2k-1}}{c_2 c_4 \ldots c_{2k}}$, $\sum_{k=1}^{\infty} \frac{c_2 c_4 \ldots c_{2k-2}}{c_1 c_3 \ldots c_{2k-1}}$ diverges.

(B) If the S-fraction (3) converges at a single point in $\mathcal{H}$, then it converges at all points in $\mathcal{H}$ to a holomorphic function.

(C) A sufficient condition for an S-fraction (3) to converge to a function holomorphic in $\mathcal{H}$ is that there exists a constant $M > 0$ such that $c_k < M$, $k \geq 1$.

2 CONVERGENCE

We give two convergence criteria for multidimensional S-fraction with independent variables (1). For use in the following theorems we introduce the notation for the tails of (1):

$$Q_{i(n)}^{(n)}(\mathbf{z}) = 1, \; i(n) \in \mathcal{I}_n, \; n \geq 1, \tag{4}$$

$$Q_{i(k)}^{(n)}(\mathbf{z}) = 1 + \sum_{i_{k+1}=1}^{i_k} \frac{c_{i(k+1)} z_{i_{k+1}}}{1} + \sum_{i_{k+2}=1}^{i_{k+1}} \frac{c_{i(k+2)} z_{i_{k+2}}}{1} + \cdots + \sum_{i_n=1}^{i_{n-1}} \frac{c_{i(n)} z_{i_n}}{1}, \tag{5}$$

where $i(k) \in \mathcal{I}_k,\ 1 \le k \le n-1,\ n \ge 2$. It is clear that the following recurrence relations hold

$$Q_{i(k)}^{(n)}(\mathbf{z}) = 1 + \sum_{i_{k+1}=1}^{i_k} \frac{c_{i(k+1)} z_{i_{k+1}}}{Q_{i(k+1)}^{(n)}(\mathbf{z})},\ i(k) \in \mathcal{I}_k,\ 1 \le k \le n-1,\ n \ge 2. \tag{6}$$

Let $f_n(\mathbf{z}) = 1 + \sum_{i_1=1}^{N} \frac{c_{i(1)} z_{i_1}}{Q_{i(1)}^{(n)}(\mathbf{z})}$ be the nth approximant of (1), $n \ge 1$.

We shall prove the following result.

Theorem 2. *A multidimensional S-fraction with independent variables (1), where the $c_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \ge 2$, satisfy the conditions*

$$\sum_{i_{k+1}=1}^{i_k} c_{i(k+1)} \le r,\ i(k) \in \mathcal{I}_k,\ k \ge 1, \tag{7}$$

where r is a positive number, converges to a function holomorphic in the domain

$$\mathcal{P}_{r,M} = \bigcup_{\alpha \in (-\pi/2, \pi/2)} \left\{ \mathbf{z} \in \mathbb{C}^N :\ |z_k| - \mathrm{Re}(z_k e^{-2i\alpha}) < \frac{\cos^2(\alpha)}{2r},\ |z_k| < M,\ 1 \le k \le N \right\} \tag{8}$$

for every constant $M > 0$. The convergence is uniform on every compact subset of $\mathcal{P}_{r,M}$.

Proof. Let α be arbitrary number from the interval $(-\pi/2, \pi/2)$ and let n be arbitrary natural number. Using relations (6), by induction on k for arbitrary of multiindex $i(k) \in \mathcal{I}_k$ we show that the following inequalities are valid

$$\mathrm{Re}(Q_{i(k)}^{(n)}(\mathbf{z}) e^{-i\alpha}) > \frac{\cos(\alpha)}{2} > 0, \tag{9}$$

where $1 \le k \le n$.

It is clear that for $k = n$, $i(n) \in \mathcal{I}_n$, relations (9) hold. By induction hypothesis that (9) hold for $k = p+1$, $p \le n-1$, $i(p+1) \in \mathcal{I}_{p+1}$, we prove (9) for $k = p$, $i(p) \in \mathcal{I}_p$. Indeed, use of relations (6) for arbitrary of multiindex $i(p) \in \mathcal{I}_p$ lead to

$$Q_{i(p)}^{(n)}(\mathbf{z}) e^{-i\alpha} = e^{-i\alpha} + \sum_{i_{p+1}=1}^{i_p} \frac{c_{i(p+1)} z_{i_{p+1}} e^{-2i\alpha}}{Q_{i(p+1)}^{(n)}(\mathbf{z}) e^{-i\alpha}}.$$

In the proof of Lemma 4.41 [9] it is shown that if $x \ge c > 0$ and $v^2 \le 4u + 4$,

$$\min_{-\infty < y < +\infty} \mathrm{Re} \frac{u + iv}{x + iy} = -\frac{\sqrt{u^2 + v^2} - u}{2x}. \tag{10}$$

We set

$$u = \mathrm{Re}(c_{i(p+1)} z_{i_{p+1}} e^{-2i\alpha}),\ v = \mathrm{Im}(c_{i(p+1)} z_{i_{p+1}} e^{-2i\alpha}),$$
$$x = \mathrm{Re}(Q_{i(p+1)}^{(n)}(\mathbf{z}) e^{-i\alpha}),\ y = \mathrm{Im}(Q_{i(p+1)}^{(n)}(\mathbf{z}) e^{-i\alpha}).$$

Then for the arbitrary index i_{p+1}, $1 \le i_{p+1} \le i_p$, it follows from (7) and (8) that

$$|c_{i(p+1)} z_{i_{p+1}} e^{-2i\alpha}| - \mathrm{Re}(c_{i(p+1)} z_{i_{p+1}} e^{-2i\alpha}) < \frac{\cos^2(\alpha)}{2}.$$

From this inequality it is easily shown that $v^2 \le 4u + 4$.

Using (6)—(10) and induction hypothesis, we obtain

$$\operatorname{Re}(Q_{i(p)}^{(n)}(\mathbf{z})e^{-i\alpha}) \ge \cos(\alpha) - \sum_{i_{p+1}=1}^{i_p} \frac{c_{i(p+1)}(|z_{i_{p+1}}| - \operatorname{Re}(z_{i_{p+1}}e^{-2i\alpha}))}{2\operatorname{Re}(Q_{i(p+1)}^{(n)}(\mathbf{z})e^{-i\alpha})}$$
$$> \cos(\alpha) - \sum_{i_{p+1}=1}^{i_p} \frac{c_{i(p+1)}\cos(\alpha)}{2r} \ge \frac{\cos(\alpha)}{2} > 0.$$

It follows from (9) that $Q_{i(k)}^{(n)}(\mathbf{z}) \not\equiv 0$ for all indices. Thus, the approximants $f_n(\mathbf{z})$, $n \ge 1$, of (1) form a sequence of functions holomorphic in $\mathcal{P}_{r,M}$.

Again, let α be arbitrary number from the interval $(-\pi/2, \pi/2)$. And, let

$$\mathcal{P}_{\alpha,\sigma,r,M} = \left\{ \mathbf{z} \in \mathbb{C}^N : \ |z_k| - \operatorname{Re}(z_k e^{-2i\alpha}) < \frac{\sigma\cos^2(\alpha)}{2r},\ |z_k| < \sigma M,\ 1 \le k \le N \right\}, \tag{11}$$

where $0 < \sigma < 1$. We set

$$c = \max_{1 \le i_1 \le N} c_{i(1)}. \tag{12}$$

Using (9), (11) and (12), for the arbitrary $\mathbf{z} \in \mathcal{P}_{\alpha,\sigma,r,M}$, $\mathcal{P}_{\alpha,\sigma,r,M} \subset \mathcal{P}_{r,M}$, we obtain for $n \ge 1$

$$|f_n(\mathbf{z})| \le 1 + \sum_{i_1=1}^{N} \frac{c_{i(1)}|z_{i_1}|}{\operatorname{Re}(Q_{i(1)}^{(n)}(\mathbf{z})e^{-i\alpha})} < 1 + \sum_{i_1=1}^{N} \frac{2c\sigma M}{\cos(\alpha)} = C(\mathcal{P}_{\alpha,\sigma,r,M}),$$

where the constant $C(\mathcal{P}_{\alpha,\sigma,r,M})$ depends only on the domain (11), i.e. the sequence $\{f_n(\mathbf{z})\}$ is uniformly bounded in $\mathcal{P}_{\alpha,\sigma,r,M}$.

Let $\mathcal{K}$ be an arbitrary compact subset of $\mathcal{P}_{r,M}$. Let us cover $\mathcal{K}$ with domains of form (11). From this cover we choose the finite subcover $\mathcal{P}_{\alpha_j,\sigma_j,r,M}$, $1 \le j \le k$.$\mathcal{P}_{\alpha_1,\sigma_1,r,M}$, $\mathcal{P}_{\alpha_2,\sigma_2,r,M}$, $\ldots$, $\mathcal{P}_{\alpha_k,\sigma_k,r,M}$. We set

$$C(\mathcal{K}) = \max_{1 \le j \le k} C(\mathcal{P}_{\alpha_j,\sigma_j,r,M}).$$

Then for arbitrary $\mathbf{z} \in \mathcal{K}$ we obtain $|f_n(\mathbf{z})| \le C(\mathcal{K})$, for $n \ge 1$, i.e. the sequence $\{f_n(\mathbf{z})\}$ is uniformly bounded on each compact subset of the domain (8).

Let $m = \max\{c, r, 1/(2MN)\}$ and let

$$\mathcal{L}_m = \left\{ \mathbf{z} \in \mathbb{R}^N : \ 0 < z_k < \frac{1}{4mN},\ 1 \le k \le N \right\}.$$

Then for the arbitrary $\mathbf{z} \in \mathcal{L}_m$, $\mathcal{L}_m \subset \mathcal{P}_{r,M}$, we obtain

$$|c_{i(1)}z_{i_1}| < \frac{c}{4mN} < \frac{1}{2N},\ |c_{i(k+1)}z_{i_{k+1}}| < \frac{r}{4mN} \le \frac{1}{4i_k},\ i(k) \in \mathcal{I}_k,\ k \ge 1.$$

It follows from Theorem 1 [4], with $g_{i(k)} = 1/2$, $i(k) \in \mathcal{I}_k$, $k \ge 1$, that (1) converges in $\mathcal{L}_m$. Hence by Theorem 24.2 [10, pp. 108—109] (see also Theorem 2.17 [3, p. 66]), the multidimensional S-fraction with independent variables (1) converges uniformly on compact subsets of $\mathcal{P}_{r,M}$ to a holomorphic function. □

The following theorem can be proved in much the same way as Theorem 2 using Theorem 4 [4].

Theorem 3. *A multidimensional S-fraction with independent variables (1), where the* $c_{i(k)}$*,* $i(k) \in \mathcal{I}_k$*,* $k \geq 2$*, satisfy the conditions* $c_{i(k)} \leq r,\ i(k) \in \mathcal{I}_k,\ k \geq 2$*, where* r *is a positive number, converges to a function holomorphic in the domain*

$$\mathcal{D}_{r,M} = \bigcup_{\alpha \in (-\pi/2, \pi/2)} \left\{ \mathbf{z} \in \mathbb{C}^N :\ \sum_{k=1}^{N} \Big(|z_k| - \mathrm{Re}(z_k e^{-2i\alpha}) \Big) < \frac{\cos^2(\alpha)}{2r},\ \sum_{k=1}^{N} |z_k| < M \right\}$$

for every constant $M > 0$*. The convergence is uniform on every compact subset of* $\mathcal{D}_{r,M}$*.*

Next, we give two convergence criteria for multidimensional S-fractions with independent variable (2). In addition to (4) and (5), for the tails of (2) we introduce the following notation:

$$Q_{i(0)}^{(0)}(\mathbf{z}) = 1,\ Q_{i(0)}^{(n)}(\mathbf{z}) = 1 + \sum_{i_1=1}^{N} \frac{c_{i(1)} z_{i_1}}{1} \,+\, \sum_{i_2=1}^{i_1} \frac{c_{i(2)} z_{i_2}}{1} \,+\, \cdots \,+\, \sum_{i_n=1}^{i_{n-1}} \frac{c_{i(n)} z_{i_n}}{1},\ n \geq 1.$$

And, thus, the nth approximant of (2) we may write as $g_n(\mathbf{z}) = 1/Q_{i(0)}^{(n-1)}(\mathbf{z})$, $n \geq 1$.

Now we shall prove the following result.

Theorem 4. *A multidimensional S-fraction with independent variables (2), where the* $c_{i(k)}$*,* $i(k) \in \mathcal{I}_k$*,* $k \geq 1$*, satisfy the conditions*

$$\sum_{i_k=1}^{i_{k-1}} c_{i(k)} \leq r,\ i(k) \in \mathcal{I}_k,\ k \geq 1, \tag{13}$$

where r *is a positive number, converges to a function holomorphic in the domain*

$$\mathcal{P}_r = \bigcup_{\alpha \in (-\pi/2, \pi/2)} \left\{ \mathbf{z} \in \mathbb{C}^N :\ |z_k| - \mathrm{Re}(z_k e^{-2i\alpha}) < \frac{\cos^2(\alpha)}{2r},\ 1 \leq k \leq N \right\}. \tag{14}$$

The convergence is uniform on every compact subset of $\mathcal{P}_r$*.*

Proof. Let α be arbitrary number from the interval $(-\pi/2, \pi/2)$. By analogy with (9) it is easy to prove the validity of the following inequalities

$$\mathrm{Re}(Q_{i(k)}^{(n-1)}(\mathbf{z}) e^{-i\alpha}) > \frac{\cos(\alpha)}{2} > 0, \tag{15}$$

where $n \geq 1$, $0 \leq k \leq n-1$, $i(k) \in \mathcal{I}_k$, if $k \geq 1$. It follows from (15) that $Q_{i(k)}^{(n-1)}(\mathbf{z}) \not\equiv 0$ for all indices. It means that the approximants $g_n(\mathbf{z})$, $n \geq 1$, of (2) form a sequence of functions holomorphic in $\mathcal{P}_r$.

Again, let α be arbitrary number from the interval $(-\pi/2, \pi/2)$. And, let

$$\mathcal{P}_{\alpha,\sigma,r} = \left\{ \mathbf{z} \in \mathbb{C}^N :\ |z_k| - \mathrm{Re}(z_k e^{-2i\alpha}) < \frac{\sigma \cos^2(\alpha)}{2r},\ 1 \leq k \leq N \right\}, \tag{16}$$

where $0 < \sigma < 1$. Using (15) for the arbitrary $\mathbf{z} \in \mathcal{P}_{\alpha,\sigma,r}$, $\mathcal{P}_{\alpha,\sigma,r} \subset \mathcal{P}_r$, we obtain for $n \geq 1$

$$|g_n(\mathbf{z})| \leq \frac{1}{\mathrm{Re}(Q_{i(0)}^{(n-1)}(\mathbf{z})e^{-i\alpha})} < \frac{2}{\cos(\alpha)} = C(\mathcal{P}_{\alpha,\sigma,r}),$$

where the constant $C(\mathcal{P}_{\alpha,\sigma,r})$ depends only on the domain (16), i.e. the sequence $\{g_n(\mathbf{z})\}$ is uniformly bounded in $\mathcal{P}_{\alpha,\sigma,r}$.

Let $\mathcal{K}$ be an arbitrary compact subset of $\mathcal{P}_r$. Let us cover $\mathcal{K}$ with domains of form (16). From this cover we choose the finite subcover $\mathcal{P}_{\alpha_1,\sigma_1,r}$, $\mathcal{P}_{\alpha_2,\sigma_2,r}$, $\ldots$, $\mathcal{P}_{\alpha_k,\sigma_k,r}$. We set $C(\mathcal{K}) = \max_{1\leq j\leq k} C(\mathcal{P}_{\alpha_j,\sigma_j,r})$. Then for arbitrary $\mathbf{z} \in \mathcal{K}$ we obtain $|g_n(\mathbf{z})| \leq C(\mathcal{K})$, for $n \geq 1$, i.e. the sequence $\{g_n(\mathbf{z})\}$ is uniformly bounded on each compact subset of the domain (14).

Let $\mathcal{L}_r = \left\{\mathbf{z} \in \mathbb{R}^N : \ 0 < z_k < \frac{1}{4rN},\ 1 \leq k \leq N\right\}$. Then from (13) for the arbitrary $\mathbf{z} \in \mathcal{L}_r$, $\mathcal{L}_r \subset \mathcal{P}_r$, we obtain

$$|c_{i(k)} z_{i_k}| < \frac{1}{4N} \leq \frac{1}{4 i_{k-1}},\ i(k) \in \mathcal{I}_k,\ k \geq 1.$$

It follows from Theorem 2 [4], with $g_{i(k)} = 1/2$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, that (2) converges in $\mathcal{L}_r$. Hence by Theorem 24.2 [10, pp. 108—109] (see also Theorem 2.17 [3, p. 66]), the multidimensional S-fraction with independent variables (2) converges uniformly on compact subsets of $\mathcal{P}_r$ to a holomorphic function. □

Finally, the following theorem can be proved in much the same way as Theorem 4 using Theorem 5 [4].

Theorem 5. *A multidimensional S-fraction with independent variables (2), where the $c_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, satisfy the conditions $c_{i(k)} \leq r,\ i(k) \in \mathcal{I}_k,\ k \geq 1$, where r is a positive number, converges to a function holomorphic in the domain*

$$\mathcal{D}_r = \bigcup_{\alpha \in (-\pi/2,\pi/2)} \left\{\mathbf{z} \in \mathbb{C}^N : \ \sum_{k=1}^{N} \Big(|z_k| - \mathrm{Re}(z_k e^{-2i\alpha})\Big) < \frac{\cos^2(\alpha)}{2r}\right\}.$$

The convergence is uniform on every compact subset of $\mathcal{D}_r$.

The following two corollaries are an immediate consequences of Theorems 2 and 4 respectively.

Corollary 1. *An S-fraction (3), where the c_k, $k \geq 2$, satisfy the conditions $c_k \leq r,\ k \geq 2$, where r is a positive number, converges to a function holomorphic in the domain*

$$\mathcal{H}_{r,M} = \left\{z \in \mathbb{C} : \ \left|\arg\left(z + \frac{1}{4r}\right)\right| < \pi,\ |z| < M\right\}$$

for every constant $M > 0$. The convergence is uniform on every compact subset of $\mathcal{H}_{r,M}$.

Corollary 2. *An S-fraction*

$$\frac{1}{1+}\frac{c_1 z}{1}\,\frac{}{+}\frac{c_2 z}{1}\,\frac{}{+}\frac{c_3 z}{1}\,\frac{}{+}\cdots,$$

where the c_k, $k \geq 1$, satisfy the conditions $c_k \leq r,\ k \geq 1$, where r is a positive number, converges to a function holomorphic in the domain

$$\mathcal{H}_r = \left\{z \in \mathbb{C} : \ \left|\arg\left(z + \frac{1}{4r}\right)\right| < \pi\right\}.$$

The convergence is uniform on every compact subset of $\mathcal{H}_r$.

We note that, in view of Theorem 1, we conclude that Corollaries 1 and 2 give us two new convergence criteria for S-fractions.

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Досліджується збіжність багатовимірних S-дробів з нерівнозначними змінними, які є багатовимірним узагальненням S-дробів. Ці гіллясті ланцюгові дроби є ефективним інструментом для наближення функцій, заданих формальними кратними степеневими рядами. Для встановлення критеріїв збіжності використовується теорема про продовження збіжності із уже відомої малої області до більшої. У результаті показано, що перетин параболічної і кругової областей є областю збіжності багатовимірного S-дробу з нерівнозначними змінними, а параболічна область є областю збіжності гіллястого ланцюгового дробу, який є оберненим до багатовимірного S-дробу з нерівнозначними змінними. Крім того, отримано два нових критерії збіжності для S-дробів як наслідки з вище згаданих результатів.

Ключові слова і фрази: збіжність, рівномірна збіжність, S-дріб, багатовимірний S-дріб з нерівнозначними змінними.